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**★The mathematical foundations of mixing.**

The linked twist map as a paradigm in applications: micro to macro, fluids to solids.
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This book follows, after almost twenty years, J. M. Ottino's *The kinematics of mixing: stretching, chaos, and transport* [Cambridge Univ. Press, Cambridge, 1989; MR1001565 (90k:58136)], which was aimed at introducing to a broad audience the basic toolbox of dynamical systems theory in the analysis of mixing in deterministic (e.g. laminar) fluid flow. Two decades of interdisciplinary studies in this line of investigation have made it clear that the formal setting for approaching the phenomenology of laminar mixing involves tools and concepts that stretch to the boundary of contemporary research in ergodic theory, starting from the theory of nonuniformly partially hyperbolic systems, developed by Ya. B. Pesin [Uspehi Mat. Nauk **32** (1977), no. 4 (196), 55–112, 287; MR0466791 (57 #6667)] in the mid-1970s, and extended a decade later to the case of maps with singularities by A. B. Katok et al. [*Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lecture Notes in Math., 1222, Springer, Berlin, 1986; MR0872698 (88k:58075)]. The framework developed by Pesin is physically appealing in that it deals with systems where chaos is "massive but not ubiquitous", which is exactly the typical behavior observed in physically interesting models, including fluid mixing systems. Thus, Pesin theory relaxes the notion of uniform hyperbolicity introduced by Smale and Anosov in the sixties [S. Smale, Bull. Amer. Math. Soc. **73** (1967), 747–817; MR0228014 (37 #3598); D. V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*, Amer. Math. Soc., Providence, R.I., 1969; MR0242194 (39 #3527)], where the hyperbolic splitting of the tangent space into contracting and dilating subspaces is assumed to hold true at any point of the tangent space, a condition that is hardly verified by nonlinear models of physical interest. A peculiar aspect of Pesin theory is that examples of nonuniformly hyperbolic systems with mixed behavior whose ergodic properties can be proved analytically (beyond numerical evidence) are extremely hard to construct. The class of Linked Twist Maps (LTMs) constitutes one such example [R. L. Devaney, Proc. Amer. Math. Soc. **71** (1978), no. 2, 334–338; MR0494289 (58 #13194); in *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, 121–145, Lecture Notes in Math., 819, Springer, Berlin, 1980; MR0591180 (82f:58070)], and the book under review proposes this class as a basic archetype for a rigorous approach to the physical notion of streamline crossing introduced in the fluid mechanical context [J. M. Ottino, op. cit.], and considered, by common wisdom, the fingerprint of efficient mixing.

Physically, the framework is still purely kinematic, where transport, stretching, and folding of passive fluid elements are primarily targeted for the quantitative assessment of mixing efficiency.

Chapter 1 provides the physical background by illustrating some key examples of mixing systems, spanning a wide order of length scales (from geophysical phenomena to microflow devices for DNA sequencing), and encompassing fluid as well as powder mixing. Chapter 2 introduces informally LTMs, first by taking the two-torus as a base manifold (Toral Linked Twist Maps, TLTMs), and then considering the planar counterparts of TLTMs, where the dynamics takes place in the union of two annuli of $\mathbb{R}^2$ that intersect each other transversally. The latter case is taken as more closely representative of fluid mixing systems in that the annuli can be associated with two streamline systems that are periodically (temporally or spatially) alternated to create the stirring protocol. Chapter 3 provides a concise introduction to ergodic theory. A constant effort is maintained to connect the abstract definitions of ergodicity, topological transitivity, topological mixing, and measure-theoretical (weak) mixing to the context of fluid mixing. Chapter 4 is entirely devoted to proving the existence of a Smale horseshoe for LTMs of the plane. After a concise introduction to the standard Smale horseshoe and to symbolic dynamics, the classical example of Devaney [op. cit.; MR0591180 (82f:58070)] is discussed. The smooth conjugacy of the family of LTMs to a subshift of finite type is proved by closely following the argument developed in [R. L. Devaney, op. cit.; MR0494289 (58 #13194)].

Chapter 5 explores modern ergodic theory and can be regarded as a succinct account of Pesin theory of nonuniform (partial) hyperbolicity. It includes the extension of the theory to the case of smooth maps with singularities developed by A. B. Katok et al. [op. cit.]. The final subsection illustrates the method of invariant cones for proving the existence of trajectories with positive Lyapunov exponents, a tool that circumvents the problem of finding pointwise the expanding and contracting directions. Chapter 6 illustrates the application of Pesin theory to TLTMs, which are here introduced formally in a general setting. The ergodic partition is proved through Pesin theory by showing that the set of trajectories with positive Lyapunov exponent has positive measure. The case of TLTM with singularities is discussed for both the co-rotating and counter-rotating cases, following the approach of M. Wojtkowski [in *Nonlinear dynamics (Internat. Conf., New York, 1979)*, 65–76, Ann. New York Acad. Sci., 357, New York Acad. Sci., New York, 1980; MR0612807 (83d:28009)].

The geometric implications of the ergodic decomposition for TLTM are analyzed in Chapter 7. The Bernoulli property is here established for co-rotating TLTMs. Conditions on twist strength and wrapping number that ensure the Bernoulli property are also given for counter-rotating TLTMs. Chapter 8 discusses the conditions on twist strength for the existence of the ergodic partition in the case of linked twist maps of the plane.

The last chapter of the book surveys the open problems and the possible directions of future research. Some critical aspects of the practical applications of the theory are discussed here, such as the identification of the annuli for constructing the planar LTM, as well as the lack of monotonicity of the functions generating the planar LTMs in concrete fluid mixing systems, where no-slip boundary conditions are imposed at the boundaries. In this respect, this reviewer thinks that the major obstacle in connecting the results obtained for planar LTMs to real-world mixing systems is the difficulty of constructing a stirring protocol where two annuli can be identified, such that the dynamics of almost all the points in the union of the annuli is described by the LTM constructed with the methods suggested by the authors. Specifically, the problem is that in an actual flow system, most of the fluid particles will eventually be mapped outside the annular domain, and therefore the results concerning the measure-theoretical properties of the LTM (such as the ergodic decomposition) need not coincide with those actually exhibited by the physical flow system.

The material is presented in a style that should make it accessible to a wide audience, and especially to readers involved in practical aspects of mixing who wish to learn more about the mathematical problems underlying the physical phenomenology. In this respect, a more comprehensive list of references pointing at work that is strictly related to the topics discussed in the book, such as the definition of invariant measures associated with the stable and unstable foliations [M. Giona and A. Adrover, Phys. Rev. Lett. **81** (1998), no. 18, 3864–3867], or the analytical results obtained by C. Liverani and R. S. MacKay in the investigation of non-monotonic TLTMs with mixed behavior, would have been useful for the reader.

Reviewed by *Stefano Cerbelli*